§4. Applications of the containment-parthood distinction. The aim of this section is to scrutinize some of the applications of the containment-parthood distinction. The fruitfulness of the distinction will provide us with an (*a posteriori*) justification of it; moreover, we shall see that in some cases Leibniz needs an infinitary Real Addition operation.

4.1. Monads are in bodies, but not parts of bodies. We have shown that, provided there are at least two objects, there can be no atom according to the LM calculus. Since an atom is defined as something with no proper part, their metaphysical counterparts are monads. In this paragraph, with Ax we shall indicate that x is a monad (or simple substance). The conclusion is that monads are not parts of any compound: in particular, monads are not parts of bodies. Indeed, monads are not homogeneous to bodies, since the former are immaterial and not extended, the latter material and extended. However, monads "enter into compounds", i.e., they are (contained) in bodies. In the Monadology we read:

> 1. The monad which we are to discuss here is nothing but a simple substance which enters into compounds. *Simple* means without parts. (*Theodicy*, Sec. 10).
> 2. There must be simple substances, since there are compounds, for the compounded is but a collection or an *aggregate* of simples. (GP VI 607 - [35, p. 643])

In the *Principles of Nature and Grace* it is clarified that these compounds are bodies:

> Compounds, or bodies [Les composés ou les corps] are pluralities, and simple substances—lives, souls, and spirits—are unities. (GP VI 598; [35, p. 636])

This latter thought was echoed in the 1690s as follows:

> A body is not a substance but an aggregate of substances. (Leibniz 1690: A VI, 4, 1668)

A body is an aggregate or a compound of simple substances. Notice that Leibniz does not say that a body is a whole whose parts are the monads; indeed, he is very clear that this is not the case:

> Moreover, even if the body of an animal or my own body is composed, in turn, of innumerble substances, they are not parts of the animal or of me. (GM III 537—translation from [34, p. 167])

That bodies are aggregates of substances means that substances are (contained) in bodies. Here Leibniz is simply applying the definition of containment of the RA calculus: if x is in y, then x plus something else is identical to y, which means that y is an aggregate (a container) of which x is one of the ingredients.

Remarkably, in order for this distinction to work, real addition must be an infinitary operation along that of RA^{∞}: each body is constituted by infinitely many substances, i.e., each is an infinite aggregate of monads. If we admitted only a finitary operation

not requiring that homogeneity preserves open intervals, since set difference among open intervals does not always yield back open sets).

(like that of RA), then we could not interpret the relationship between monads and bodies via the containment relation, and so it would become totally mysterious which relations are here in play.[54]

Working within RA^∞, we can be more specific on the relationship among bodies and monads. The gist of Leibniz's theory can be summed up in 6 propositions:

1. There are monads, i.e., substances with no proper parts: in formulas, $\exists x Ax$;
2. There are bodies, i.e., concrete objects with proper parts: $\exists x Bx$, where $Bx \equiv Conc(x) \wedge \exists w PPwx$ ($Conc(x)$ means the x is concrete, i.e., not ideal or abstract);
3. What exists is either a monad or a body;
4. Each part of a body is a body as well, and so it has its own proper parts: $\forall x(Bx \rightarrow \forall y(PPyx \rightarrow \neg Ay))$;
5. There are monads in bodies: $\forall x(Bx \rightarrow \exists y(Ay \wedge Cxy))$;
6. Bodies are aggregate of monads (only), i.e., bodies results from monads only: $\forall x(Bx \rightarrow \Sigma_\phi x)$, where ϕ is the condition $Ay \wedge Cxy$.

Propositions 1 and 2 simply state the existence of, respectively, bodies and monads. Proposition 3 states that if something exists, either it is a monad or a body. Together with Proposition 6, this entails that what exists is either a monad or an aggregate of monads. Proposition 4 follows from PP-Non-Well-Foundedness applied to bodies: since bodies have parts, by PP-Non-Well-Foundedness, these parts have further parts and so on. So no atom (i.e., no monad) is a part of a body. Proposition 5 states that monads are contained in bodies. Notice that it does not explicitly say that all that there is to bodies are monads. This latter claim is in fact Proposition 6, which says that bodies are aggregates of monads. In other words, bodies are the sum (real addition) of all and only the monads they contain. So, if b is a body, then b is the sum of all monads y (Ay) contained in it (Cby). By applying UCP (where ϕ is the condition $Ax \wedge Cbx$) we obtain: $B(b) \rightarrow \Sigma_\phi b$. 6 is the generalization of this. Given Leibniz's nominalistic attitude toward aggregates (and sums) this amounts to the idea that the reality of bodies just is the reality of the monads in them.

At this point one might worry about the mutual consistency of propositions 4 and 6. The latter claims that bodies are aggregates of monads only: the idea is that we do not need anything else apart from monads to obtain bodies; when God creates a body he simply created the correspondent monads. But Proposition 4 says that there are parts in bodies, and we know that these parts are not monads.

To meet this concern the first thing to notice is that, given the RA^∞ calculus, Propositions 5 and 6 are equivalent.

Theorem: 5 $\leftrightarrow$ 6

Proof: From right to left. That 6 implies 5 is straightforward: if $B(b)$ is the case for an arbitrary b, then $\Sigma_\phi b$ is the sum of all monads in b. In order for the sum to exist, ϕ must be satisfied, i.e., there must be a y such a that Ay and Cby. Therefore, we have $B(b) \rightarrow \exists y(Ay \wedge Cby)$. Generalizing we obtain 5.

From left to right. The other direction is more complex, and here we adopt (a slightly modified version of) a proof from [17, p. 146]. First, notice that in virtue of the definition of $\Sigma_\phi x$ proposition 6 amounts to the conjunction of the following two propositions:

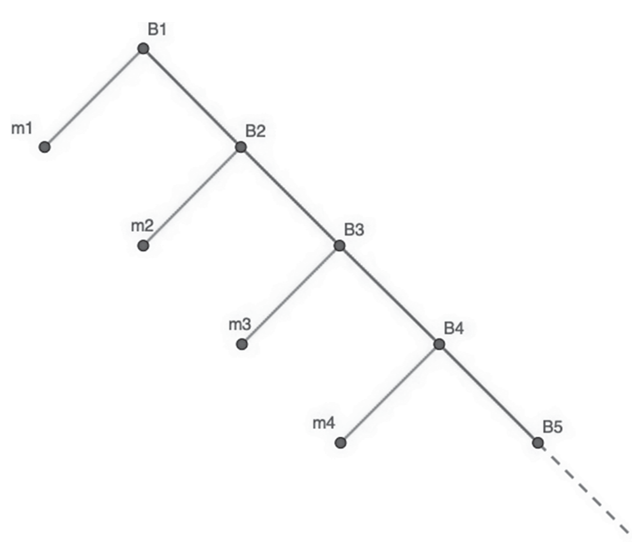
[54] For the relationship between monads and bodies see the illuminating pages of [4].

6A) $\forall x(Bx \to (\forall z(Az \wedge Cxz) \to Cxz))$,
6B) $\forall x(Bx \to \forall y(\forall z(Az \wedge Cxz) \to Cyz) \to Cyx)$.
Since 6A is a logical truth, it is enough to show that 5 implies 6B. To do that we shall use our Subtraction 2 axiom, according to which, given any x and y, $x - y$ is always defined as the only object z such that Cxz and Dzy (when Cyx or $x = y$, the result is *Nihil*). The proof goes as follows.

Suppose that $B(a)$. Moreover, suppose that $\neg C(b, a)$. By Subtraction 2, there is a z such that $\forall w(C(z, w) \leftrightarrow (C(a, w) \wedge D(w, b)))$. Thus $z = a - b$. In particular, we have that there is a w such that $C(a, w) \wedge D(w, b)$. By Proposition 3, w is a monad or a body. If it is a body, by Proposition 5, it contains a monad d (and so Ad). If it is a monad, then Aw. In this case we can put $w = d$. In each case, we have that there is a d such that Ad and $C(w, d)$. By transitivity of containment, we obtain $C(a, d)$. We thus have $Ad \wedge C(a, d)$.

From $D(w, b)$ and $C(w, d)$, we get $\neg C(b, d)$. Therefore, we have $Ad \wedge C(a, d) \wedge \neg C(b, d)$. By existential generalization, we obtain $\exists z(Az \wedge C(a, z) \wedge \neg C(b, z))$. Since this depends on having assumed $\neg C(b, a)$, we can introduce the implication: $\neg C(b, a) \to \exists z(Az \wedge C(a, z) \wedge \neg C(b, z))$. Contraposing and by propositional equivalences, this amounts to $\forall z(Az \wedge C(a, z) \to C(b, z)) \to C(b, a)$. By introducing the implication and generalization, we obtain $\forall x(Bx \to \forall y(\forall z(Az \wedge C(x, z) \to C(y, z)) \to C(y, x)))$, i.e., 6B. □

The equivalence of Propositions 5 and 6 means that we just need to assume that there are monads in bodies to derive that bodies are nothing else than (non-mereological) aggregates of monads. We are now in a position to see why Propositions 4 and 6 are consistent. The following model clearly validates Propositions 4 and 5, but since 5 is equivalent to 6, it validates 6 as well.



In the picture, the lines represent the containment relation. So, $m1$ is contained in $B1$, $m2$ is contained in $B2$, and $B2$ is contained in $B1$, etc. $m1, m2$, etc. represent monads, while $B1, B2$, etc. represent bodies. The lines connecting the B-terms with the

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m-terms indicate that the containment relation holds among non-homogeneous things, while the lines connecting the B-terms with other B-terms are for containment between homogeneous things, i.e., parthood. Proposition 4 holds, because all (proper) parts of a body ($B1$, $B2$, etc.) are not atoms, namely they have further (proper) parts. At the same time Proposition 5 holds, since each body has at least one monad contained in it. But then also Proposition 6 holds, and indeed every object in the model is ultimately decomposable into monads. In particular, the sum of all monads is identical to $B1$.[55]

It can be easily seen that the present model is not a model of LM or RA^{∞}. For example, there is no sum of $m1$ and $m2$, which implies that there is no remainder of $B1 - B3$. And this may be seen as problematic, since the existence of remainders is guaranteed by Subtraction 2, which we have used in the proof of the equivalence of 5 and 6. But a closer look at the proof will reveal that we have not used all the strength of Subtraction 2. In fact the proof only relies on a weaker principle, i.e., $\neg C(y, x) \rightarrow (\exists z C(x, z) \wedge D(z, y))$. And this principle is valid in the model above. The point of the model is to show that the two claims at the heart of Leibniz's theory of substances (bodies are aggregates of substances and at the same time they are divisible without end into further composite bodies) are consistent with each other. The model can thus be seen as an oversimplified picture of the structure of a Leibnizian corporeal substance. Clearly the distinction containment-parthood is essential to it.

4.2. Multiplicity in the simple. In the Monadology, Leibniz famously claimed that there is a multiplicity (of states) in each simple substance:

> 12. But besides the principle of change there must be a particular detail of what changes, which constitutes the specific nature and the variety, so to speak, of simple substances.
> 13. This detail must enfold a multitude in the unity or the simple. For every natural change takes place by degrees—something changes and something remains—and as a result there must be a plurality of affections and of relations in the simple substance, even though it has no parts. (translation from [35, p. 644], slightly modified).

Leibniz here introduces a multiplicity of different states within each monad; multiplicity which is required for the explanation of change. This is Leibniz's reinterpretation of the traditional claim that substances have a plurality of accidents. But how is it possible that what is simple, i.e., what has no parts, hosts a multiplicity in itself? Clearly, these states cannot be parts of the substance, otherwise we would have a contradiction with the attribute of simplicity. Here, again, the distinction between containment and parthood solves the problem. States are (contained) in the substance, but are not part of it. Substances are simple with regard to the parthood relation: on the contrary, with regard to the containment relation, the substance is something complex, exactly in the sense that there is a multiplicity of states in it. Each state is in the substance, which means that in the substance there is an aggregate of states.

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55 An easy way to see that this is the case is to consider an example from [21, p. 75] (who first introduced this model). Interpret the lines as representing the subset relation; then take $B1$ to be the set of positive integers {1,2,3,...}; $B2$ the set {2,3,4,...}; $B3$ the set {3,4,5,...} etc. Moreover take $m1$ to be {1}; $m2$ {2}, etc. Then it is clear that the set $B1$ is the union of all elements $m1$, $m2$, etc.

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It goes without saying that this multiplicity of states is infinite. Here again we find an example of an (implicit) use of an infinitary Real Addition operation. But there is more: each state represents the entire world, which—as we are going to see—is an infinite aggregate of substances. Therefore, not only is there an infinite aggregate of states in each substance, but each one of these states is an infinite aggregate of representations (of everything that exists). Again, with only a finitary Real Addition, we could not apply the containment relation, and the relation in play here would be mysterious. These applications both require the RA^{∞} calculus.

4.3. The world. The last application of the containment-parthood distinction that I want to scrutinize is Leibniz's conception of the created world. Here again we find what appear to be incompatible statements about it. On the one hand, Leibniz is clear that the world constitutes an actual infinite:

> Created things are actually infinite. For any body whatever is actually divided into several parts, since any body whatever is acted upon by other bodies. And any part whatever of a body is a body by the very definition of body. So bodies are actually infinite, i.e., more bodies can be found than there are unities in any given number. (A VI 4, 1393/ [25, p. 235])
>
> There is an infinity of creatures in the smallest particle of matter, because of the actual division of the continuum to infinity. (GP VI 232/ [33, §195])

Since there is an actual infinity of bodies within each body, the world, i.e., the aggregate of every created thing, is an actual infinite as well. The world is sometimes described as the aggregate of all bodies ("The aggregate of all bodies is called the world"—A VI 4, 1509), other times it is described as the aggregate of all (created) substances (since what really exists are substances). In both cases, the infinite in play is actual, showing once again that Leibniz was using an infinitary real addition operation (indeed, from the claim that the world is the aggregate of all bodies or all substances, by the definition of *inesse* we can infer that each body and each substance is (contained) in the world).

However, there are other passages that suggest what may appear as a rather different view:

> Thus, we may indeed call all bodies together "the world", but in reality the world is not some one thing, but this alone can be said: for any given body, there is some larger one in the world and we never reach a finite body that includes all [bodies]. Nor, however, is there such an infinite body. (A VI 4, 1469 - about 1683-85).
>
> Yet M. Descartes and his followers, in making the world out to be indefinite so that we cannot conceive of any end to it, have said that matter has no limits. They have some reason for replacing the term 'infinite' by 'indefinite', for there is never an infinite whole in the world, though there are always wholes greater than others *ad infinitum*. As I have shown elsewhere, the universe itself cannot be considered to be a whole. (A VI, 6, 151/ translation from [36, pp. 150–151],)

One may be tempted to interpret these claims as suggesting the view that the world constitutes a potential infinite (given finitely many bodies, always more can be founded). But this would immediately give us a contradiction with the claim that the world is an actually infinite aggregate. Something cannot be at the same time and under the same respect potential and actual.

Again, the contradiction can be solved by appealing to the distinction between containment and parthood. When we look at the world with the "glasses" of the containment relation, the world is an infinite aggregate, but it is not an infinite whole ("infinity itself is nothing, i.e., that it is not one and not a whole"—A VI 3, 168/[25, p. 9]). Recall the nominalistic reading of aggregates: to say that the world is an infinite aggregate of substances/bodies simply means that there are actually infinitely many substances/bodies. However, when we look at it with the "glasses" of the parthood relation, what we see are finite wholes (here finite means that they have a finite magnitude), and given a whole a more comprehensive whole can always be found. Notice that the actual infinity of the universe is what guarantees that "there are always wholes greater than other ad infinitum". The idea is that FCP allows us to add together more and more things into bigger and bigger wholes. Suppose that A and B are two different things, possibly overlapping, but such that one is not contained in the other. Let us compose them via FCP into a whole C. Then A and B will both be proper parts of C: in virtue of the definition of the $<$-relation, since each of them are proper parts of C, they both are smaller than C. If we now compose C with a further thing D, we will obtain a bigger whole E. It seems that in this way, by applying FCP repeatedly, we can get bigger and bigger wholes. This only holds if one of the two things that we sum is not contained in the other. But what guarantees that, for any thing a, we can find a b, not contained in a? It is the fact that the universe is infinite that guarantees that the application of FCP can always result in bigger and bigger wholes. Because if we have infinitely many disjoint things at disposal, then given a certain thing a, there is always a distinct thing b (not contained in a) such that their composition results in a whole bigger than each of them. The picture is thus as follows: the universe is an actual infinity of substances, but not a whole.[56] There are only finite wholes, and given an arbitrary whole, there is always a bigger whole that can be obtained via FCP. This implies that Leibniz's mereology is not only gunky, but junky as well, meaning that each whole is a proper part of another whole.

§5. Conclusion. In this paper we presented a logical reconstruction of both Leibniz's Real Addition and his mereological calculus. Compared to other reconstructions of the former (such as [39, 52]), we interpreted the Real Addition calculus not as pertaining to logic conceived as a theory of (valid) inferences, but as a "mereological" theory (here I am using the adjective 'mereological' as referring to contemporary mereology). In doing this we followed the suggestion of [44].

A consequence of this interpretation is that the Real Addition calculus—RA (or RA^{∞})—corresponds to what we nowadays call mereology. When we think of mereology as a general calculus of individuals that can be applied to any object whatsoever, then, in Leibniz's own terms, we are dealing with the Real Addition

[56] On the metaphysical consequences of the claim that the world is not a whole see [15].